Abstract—An exact and explicit solution to a class of degaussing problems is given. The solution is shown to be unique. The stability of the solution is also analyzed.

Index Terms—Degaussing, exact solution, magnetostatics.

I. INTRODUCTION

When placed in the earth’s magnetic field, an object with pronounced ferromagnetic properties produces a perturbation to the exterior magnetic field. Since the perturbation can be utilized for detecting the presence of the object, a problem of practical importance is to minimize this perturbation. Ferromagnetic objects of interest can be, e.g., ships, submarines, tanks, trucks, and other vehicles.

In the case of a submarine, the reduction of the perturbation is usually achieved by placing many electrical degaussing coils (with an appropriate distribution of an applied current) on (or just inside) the surface of the submarine. Various numerical methods, such as the boundary element methods (BEM), the method of fundamental solutions (MFS), and the finite element methods (FEM), have been used to solve the degaussing problem; see e.g., [1]–[4]. In this paper, we give an exact and explicit solution to a class of degaussing problems and point out that the solution is unique.

The degaussing problem is a special type of inverse source problem in which the deviation in a magnetic field, due to the presence of a ferromagnetic object, is reduced by finding an appropriate applied surface current. Most inverse problems are ill-posed (see e.g., [5]), however, the degaussing problem is well-posed as shown in this paper.

II. THE EXACT SOLUTION AND THE UNIQUENESS THEOREM

Consider an object with a finite volume \( D \) (enclosed by its surface \( S \)) submerged in the earth’s magnetic field density \( \mathbf{B}_{\text{earth}}(\mathbf{r}) \) (note that \( \mathbf{B}_{\text{earth}} \) is the magnetic field in the absence of the object). We do not consider any hysteresis effects in this paper. Furthermore, we do not consider the magnetic disturbance due to the internal currents induced by the movement of the object in the earth’s magnetic field, as well as the disturbance due to external currents (in e.g., ocean water) sustained by some chemical potential difference on the surface of the object. Thus, the object is simply characterized by a linear relative permeability \( \mu_r \) in our model. In a realistic application, the object may consist of many compartments with different magnetic permeabilities. One can approximate such a complicated object as a homogeneous object with a constant relative permeability \( \mu_r \) (i.e., an equivalent permeability of mixed air and thin or massive ferromagnetic materials), or, several piecewise homogeneous blocks (as we show below, the present results can be easily generalized to the case when the object is piecewise homogeneous). Such an approximation is possible because we deal with static magnetic fields and we can introduce an equivalent macroscopic permeability (which is independent of the boundary condition) for a compartment object by, e.g., homogenization theory [6], [7]. One can also determine the equivalent permeability by experimental means (see e.g., [8]).

When the ferromagnetic object becomes magnetized by the earth’s magnetic field, the disturbance in the field may be thought of as arising from an equivalent surface current \( \mathbf{J}_{\text{ins}} = \mathbf{M} \times \mathbf{n} \) [9], in free space, where \( \mathbf{M} \) is the magnetization vector and \( \mathbf{n} \) is the unit outward normal vector on the boundary surface \( S \). In the submarine application, magnetic field sensors positioned in the ocean can detect the disturbance. In order to make the submarine “invisible” to the magnetic field sensors, one applies a free current distribution \( \mathbf{J}_a(\mathbf{r}) \) on the surface \( S \) of the submarine. Since the magnetization \( \mathbf{M} \) depends on the total magnetic field, it follows that \( \mathbf{J}_{\text{ins}} \) depends on both \( \mathbf{B}_{\text{earth}} \) and \( \mathbf{J}_a \). In a practical situation, the continuously distributed surface current \( \mathbf{J}_a \) must be approximated with a set of line currents, forming a mesh on the surface, by placing many large coils around the object (see e.g., [10]). In order to approximate accurately various forms of \( \mathbf{J}_a \) (when the orientation of the object varies in the earth’s magnetic field), the mesh should be dense enough and the currents in the coils should be adjustable.

A. The Degaussing Problem

Find an applied surface current distribution \( \mathbf{J}_a(\mathbf{r}) \) such that the external magnetic flux density \( \mathbf{B}_{\text{ext}}(\mathbf{r}) \) equals the unperturbed earth’s magnetic flux density \( \mathbf{B}_{\text{earth}}(\mathbf{r}) \) at every point outside the object, i.e.

\[
\mathbf{B}_{\text{ext}}(\mathbf{r}) = \mathbf{B}_{\text{earth}}(\mathbf{r}), \quad \mathbf{r} \text{ outside } D. \tag{1}
\]

The earth’s magnetic flux density \( \mathbf{B}_{\text{earth}}(\mathbf{r}) \) can be separated from the total magnetic flux density \( \mathbf{B}(\mathbf{r}) \) through the following:

\[
\mathbf{B}_{\text{earth}} \mathbf{r} = -\int_{\Sigma_{\text{outer}}} \left( (\mathbf{\hat{n}}' \cdot \mathbf{B}(\mathbf{r}')) \nabla' G(\mathbf{r}, \mathbf{r}') + (\mathbf{\hat{n}}' \times \mathbf{B}(\mathbf{r}')) \times \nabla' G(\mathbf{r}, \mathbf{r}') \right) dS' \tag{2}
\]
where $S_{\text{meas}}$ is a measurement surface located exterior to the object and where the point of evaluation $\mathbf{r}$ is located inside $S_{\text{meas}}$. $G(\mathbf{r}, \mathbf{r}')$ is the free space static Green’s function. In the three-dimensional (3-D) case we have $G(\mathbf{r}, \mathbf{r}') = (4\pi |\mathbf{r} - \mathbf{r}'|^2)^{-1}$.

In this paper, we give an explicit expression for the exact solution to the above degaussing problem. The explicit exact solution is surprisingly simple.

**Theorem 1:** The solution to the degaussing problem is unique, and the solution is given by the following explicit formula:

$$J_a(\mathbf{r}) = \frac{1}{\mu_0} \left(1 - \frac{1}{\mu_r}\right) \hat{n} \times \mathbf{B}_{\text{earth}}(\mathbf{r})$$

$$= J_a^{(0)}(\mathbf{r}), \quad \mathbf{r} \in S.$$  \hfill (3)

**Proof:** Assume that a complete degaussing (i.e., $\mathbf{B}_{\text{ext}}(\mathbf{r}) = \mathbf{B}_{\text{earth}}(\mathbf{r})$ for all $\mathbf{r} \notin D$) can be achieved by applying a certain free surface current $J_a(\mathbf{r})$. Let $\mathbf{B}_{\text{int}}$ denote the magnetic flux density in the interior region $D$. Then the continuity of the normal component of the magnetic flux density yields $\hat{n} \cdot \mathbf{B}_{\text{int}} = \hat{n} \cdot \mathbf{B}_{\text{earth}}$ at the boundary $S$. The interior region is homogeneous and free of sources, which implies $\nabla \times \mathbf{B}_{\text{int}} = 0$. A scalar potential can then be defined through $\mathbf{B}_{\text{int}} = -\nabla \phi$. Thus, one obtains from $\nabla \cdot \mathbf{B}_{\text{ext}} = 0$ and the boundary condition that

$$\nabla^2 \psi = 0, \quad \frac{\partial \psi}{\partial n} = -\hat{n} \cdot \mathbf{B}_{\text{earth}}.$$  \hfill (4)

It is well known that the Laplace equation with prescribed normal derivative at the boundary can be uniquely solved up to an additive constant. Thus, $\mathbf{B}_{\text{int}}(\mathbf{r}) = -\nabla \psi$ is unique when an exact degaussing is achieved. On the other hand, the corresponding potential for the earth’s magnetic flux density $\mathbf{B}_{\text{earth}}(\mathbf{r})$ in the absence of the object also satisfies the system (4) of equations. Therefore, the unique interior magnetic flux density must be $\mathbf{B}_{\text{int}}(\mathbf{r}) = \mathbf{B}_{\text{earth}}(\mathbf{r})$ when exact degaussing is achieved. The interior magnetic field $\mathbf{H}_{\text{int}}$, however, suffers a reduction by a factor $\mu_r$ due to the magnetic property of the interior of the object. Thus the applied surface current $J_a$ for a complete degaussing is uniquely determined by

$$J_a = \hat{n} \times (\mathbf{H}_{\text{ext}} - \mathbf{H}_{\text{int}})$$

$$= \hat{n} \times \left(\frac{1}{\mu_0} \mathbf{B}_{\text{ext}} - \frac{1}{\mu_0 \mu_r} \mathbf{B}_{\text{int}}\right)$$

$$= \frac{1}{\mu_0} \left(1 - \frac{1}{\mu_r}\right) \hat{n} \times \mathbf{B}_{\text{earth}} \quad \mathbf{r} \in S.$$  \hfill (5)

From the relation between the total surface current ($J_a + J_{\text{ms}}$) and the discontinuity in the tangential component of the magnetic flux density one obtains $J_a + J_{\text{ms}} = (1/\mu_0) \hat{n} \times (\mathbf{B}_{\text{earth}} - \mathbf{B}_{\text{int}}) = 0$, which implies (note that in the presence of an applied surface current the magnetization current $J_{\text{ms}}$ depends on both the applied surface current $J_a$ and the earth’s magnetic flux density $\mathbf{B}_{\text{earth}}$) that when complete degaussing is achieved, the magnetization current $J_{\text{ms}}$ has the same amplitude but opposite direction as the applied surface current $J_a$, i.e.,

$$J_{\text{ms}}(\mathbf{r}) = -J_a(\mathbf{r}), \quad \forall \mathbf{r} \in S.$$  \hfill (5)

All the results presented in this paper are valid for both the two-dimensional (2-D) and 3-D cases. However, for simplicity of analysis, we consider the 2-D case hereafter. In the 2-D case, one assumes that the object is an infinitely long cylinder with the axis in the $z$ direction. The present results can be easily generalized to the case when the object is piecewise homogeneous (the magnetic flux density is identical to the original earth’s magnetic flux density at every point inside or outside the object when the complete degaussing is achieved; the applied current will then be on the boundaries of each homogeneous subdomain).

Using the explicit degaussing solution (3), one can easily plot the applied surface currents for various shapes of the object. Note that $\mathbf{B}_{\text{earth}}$ in (3) is calculated by (2). We illustrate the applied surface currents in Fig. 1 for cylinders of various cross sections, namely, a $1 \times 1$ rectangular cylinder, a circular cylinder of radius $1$ m, and an elliptic cylinder with a $1.5$ m major axis and a $1$ m minor axis [see Fig. 1(a)]. We choose the parameters to be the same as the ones used for [3, Figs. 3–6], i.e., the relative permeability of the cylinder is chosen to

Fig. 1. (a) Various cross sections of the cylinders.
be \( \mu_r = 0 \), and the earth’s magnetic flux density is along the \( y \) direction with an amplitude of \( 10^{-4} \) T. The amplitudes of the applied surface current (along the \( z \) direction) for a complete degaussing are presented in Fig. 1(b)–(d) for the rectangular, circular, and elliptic cylinders, respectively, [the horizontal axis is the arc length from points \( a \rightarrow b \rightarrow c \rightarrow d \) as indicated in Fig. 1(a)]. The amplitude \( J_a \) illustrated in Fig. 1(c) for the circular cylinder has a similar shape (with different sign) as the magnetization surface current amplitude \( J_{\text{ms}} \) (in the absence of any applied surface current) given in [3, Fig. 5]. This is true for the case of a circular cylinder, as shown in the Appendix. However, it is not true in general and thus the numerical scaling method suggested in [4] is questionable in other cases.

III. THE STABILITY OF THE EXACT SOLUTION

It is of interest to know the stability properties of the exact degaussing solution given by (3), e.g., whether an applied surface current \( J_a(\mathbf{r}) \) which gives a small perturbation of the earth’s magnetic field in the exterior region will lie in the neighborhood (in some suitable sense) of the exact degaussing solution \( J_{a(0)}(\mathbf{r}) \).

In the 2-D case, we assume that the earth’s magnetic field is in the \( xy \) plane. Both the applied surface current \( J_a \) and the magnetization surface current \( J_{\text{ms}} \) are along the \( z \) direction. We write \( J_a = J_a e_z \) and \( J_{\text{ms}} = J_{\text{ms}} e_z \), where \( e_z \) is the unit vector along the \( z \) direction. Then the magnetic flux density in the exterior and interior regions are given by the following (see e.g., [3] and [4]):

\[
B_{\text{ext}}(\mathbf{r}) = B_{\text{ext(1)}}(\mathbf{r}) + \nabla \times \int_S [J_a(\mathbf{r}') + J_{\text{ms}}(\mathbf{r}')] d\mathbf{l}'
\]

\[
B_{\text{int}}(\mathbf{r}) = B_{\text{int(1)}}(\mathbf{r}) + \nabla \times \int_S [J_a(\mathbf{r}') + J_{\text{ms}}(\mathbf{r}')] d\mathbf{l}'
\]

where the Green function \( G(\mathbf{r}, \mathbf{r}') = -(1/2\pi) \ln |r - r'|/|r_0 - r'| \) (\( r_0 \) is a reference point where \( G(\mathbf{r}, \mathbf{r}') \) vanishes). When \( \mathbf{r} \)
approaches the boundary $S$, the above two equations give (see e.g., [4] and [11])

$$
\mathbf{e}_z \cdot [\mathbf{n} \times \mathbf{B}_{\text{exx}}(\mathbf{r})] = \mathbf{e}_z \cdot [\mathbf{n} \times \mathbf{B}_{\text{earth}}(\mathbf{r})] + \frac{\mu_0}{2\pi} \int_S \left[ J_a(\mathbf{r}') + J_{\text{ins}}(\mathbf{r}') \right] \frac{\cos \theta}{|\mathbf{r} - \mathbf{r}'|} \, d\mathbf{l}' + \frac{\mu_0}{2} \left[ J_a(\mathbf{r}) + J_{\text{ins}}(\mathbf{r}) \right], \quad \mathbf{r} \to S
$$

(8)

where $\mu_0 \cos \theta = \mathbf{n} \cdot (\mathbf{r} - \mathbf{r}')/|\mathbf{r} - \mathbf{r}'|$, and $\mathbf{j}$ denotes the principal value of the integral. The magnetization surface current $J_{\text{ins}}(\mathbf{r})$ is determined by the following boundary condition:

$$
\mathbf{n} \times \left[ \frac{1}{\mu_0} \mathbf{B}_{\text{exx}}(\mathbf{r}) - \frac{1}{\mu_0 \mu_r} \mathbf{B}_{\text{earth}}(\mathbf{r}) \right] = \mathbf{J}_a, \quad \mathbf{r} \in S
$$

which gives [cf. (8) and (9)]

$$
\frac{\mu_0 + 1}{2\mu_r} J_{\text{ins}}(\mathbf{r}) + \frac{\mu_0 - 1}{2\mu_r} \int_S J_{\text{ins}}(\mathbf{r}') \frac{\cos \theta}{|\mathbf{r} - \mathbf{r}'|} \, d\mathbf{l}' = \frac{\mu_0 - 1}{2\mu_r} J_a(\mathbf{r}) - \frac{\mu_0 - 1}{2\mu_0 \mu_r} \int_S J_a(\mathbf{r}') \frac{\cos \theta}{|\mathbf{r} - \mathbf{r}'|} \, d\mathbf{l}'
$$

$$
- \frac{1}{\mu_0} \left( 1 - \frac{1}{\mu_r} \right) \mathbf{e}_z \cdot [\mathbf{n} \times \mathbf{B}_{\text{earth}}(\mathbf{r})].
$$

(9)

The above Fredholm integral equation of the second kind gives the unique solution for $J_{\text{ins}}$ when both $\mathbf{B}_{\text{earth}}(\mathbf{r})$ and $\mathbf{J}_a$ are known. One can easily check that the exact degaussing solution

$$
J_a^{(0)}(\mathbf{r}) = - J_{\text{ins}}^{(0)}(\mathbf{r})
$$

$$
= \frac{1}{\mu_0} \left( 1 - \frac{1}{\mu_r} \right) \mathbf{e}_z \cdot [\mathbf{n} \times \mathbf{B}_{\text{earth}}(\mathbf{r})]
$$

(10)

satisfies the Fredholm integral equation (10).

Theorem 2: The external magnetic flux density $\mathbf{B}_{\text{exx}}$ given by (6) is stable against small deviations of $\mathbf{J}_a$ from the exact degaussing solution $\mathbf{J}_a^{(0)}$ given by (3).

Proof: Assume that $\mathbf{J}_a(\mathbf{r}) = J_a^{(0)}(\mathbf{r}) + \varepsilon(\mathbf{r})$ and $\mu_r \leq c = \mu_M$, where $\mu_M$ is a small constant. Writing $J_{\text{ins}}(\mathbf{r}) = J_{\text{ins}}^{(0)}(\mathbf{r}) + J_{\text{ins}}^{(1)}(\mathbf{r})$, one obtains from (10) that

$$
\frac{\mu_0 + 1}{2\mu_r} J_{\text{ins}}^{(0)}(\mathbf{r}) + \frac{\mu_0 - 1}{2\mu_r} \int_S J_{\text{ins}}^{(0)}(\mathbf{r}') \frac{\cos \theta}{|\mathbf{r} - \mathbf{r}'|} \, d\mathbf{l}' = \frac{\mu_0 - 1}{2\mu_r} J_a^{(0)}(\mathbf{r}) - \frac{\mu_0 - 1}{2\mu_0 \mu_r} \int_S J_a^{(0)}(\mathbf{r}') \frac{\cos \theta}{|\mathbf{r} - \mathbf{r}'|} \, d\mathbf{l}'
$$

$$
- \frac{1}{\mu_0} \left( 1 - \frac{1}{\mu_r} \right) \mathbf{e}_z \cdot [\mathbf{n} \times \mathbf{B}_{\text{earth}}(\mathbf{r})].
$$

(11)

Eliminating the integral terms of the above two equations, one obtains

$$
J_{\text{ins}}^{(1)}(\mathbf{r}') = (\mu_r - 1) J_a^{(0)}(\mathbf{r}) - \frac{\mu_r - 1}{\mu_0} \varepsilon(\mathbf{r}).
$$

Substituting the above equation into (14), one obtains

$$
\frac{\mu_0 + 1}{2\mu_r} J_{\text{ins}}^{(1)}(\mathbf{r}) + \frac{\mu_0 - 1}{2\mu_r} \int_S J_{\text{ins}}^{(1)}(\mathbf{r}') \frac{\cos \theta}{|\mathbf{r} - \mathbf{r}'|} \, d\mathbf{l}' = \frac{\mu_0 - 1}{2\mu_r} J_a^{(1)}(\mathbf{r}) - \frac{\mu_0 - 1}{2\mu_0 \mu_r} \int_S J_{\text{ins}}^{(1)}(\mathbf{r}') \frac{\cos \theta}{|\mathbf{r} - \mathbf{r}'|} \, d\mathbf{l}'
$$

$$
- \frac{1}{\mu_0} \left( 1 - \frac{1}{\mu_r} \right) \mathbf{e}_z \cdot [\mathbf{n} \times \mathbf{B}_{\text{earth}}(\mathbf{r})].
$$

(12)

Theorem 3: The applied surface current $\mathbf{J}_a(\mathbf{r})$ is stable against small perturbations of the earth’s magnetic flux density.

Proof: Assume that $\mathbf{J}_a(\mathbf{r}) = \mathbf{J}_a^{(0)}(\mathbf{r}) + \varepsilon(\mathbf{r})$ for $\mathbf{r} \in S$, and $\mu_r \leq c = \mu_M$, where $\mu_M$ is a small constant. Writing $J_{\text{ins}}(\mathbf{r}) = J_{\text{ins}}^{(0)}(\mathbf{r}) + J_{\text{ins}}^{(1)}(\mathbf{r})$, one obtains from (9) (and 10) that

$$
\frac{1}{\mu_0} \int_S [J_a^{(1)}(\mathbf{r}') + J_{\text{ins}}^{(1)}(\mathbf{r}')] \cos \theta \frac{\mathbf{e}_z \cdot \mathbf{G}(\mathbf{r}, \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, d\mathbf{l}'
$$

$$
= \frac{1}{\mu_0} \varepsilon(\mathbf{r}).
$$

(13)

A PPENDIX

AS IMPLI E X E M PLE : T HE M AGNETIZA TION OF A C IRCULAR CYLINDER

Consider the explicit solution for the magnetization (when there is no applied surface current, i.e., $\mathbf{J}_a = \mathbf{0}$) of a circular cylinder (with a radius $a$) in the earth’s magnetic field. Assume
that the earth’s magnetic field is along the $e_y$ direction as well as the interior magnetic field after the magnetization. To the external magnetic field, the magnetized cylinder behaves like a $y$-directed dipole. Then in the cylindrical coordinate system $(\rho, \phi, z)$ one has

$$B_{\text{int}} = B_{\text{int}} e_y$$
$$= B_{\text{int}} (\sin \phi e_\rho + \cos \phi e_\phi)$$  \hspace{1cm} (A1)

$$B_{\text{ext}} = B_{\text{earth}} e_y + \frac{C}{\rho^2} (\sin \phi e_\rho - \cos \phi e_\phi)$$  \hspace{1cm} (A2)

where the constants $B_{\text{int}}$ and $C$ are to be determined. From the continuity of the $e_\rho$ component of the $B$ field and the $e_\phi$ component of the $H$ field at the surface $\rho = a$, one obtains

$$B_{\text{int}} = \frac{2\mu \rho}{\mu + 1} B_{\text{earth}},$$  \hspace{1cm} (A3)

$$C = \frac{\mu - 1}{\mu + 1} \alpha B_{\text{earth}}.$$  \hspace{1cm} (A4)

Thus the induced surface current $J_{\text{sm}}$ is determined by

$$J_{\text{sm}} = \frac{1}{\mu_0} e_\rho \times (B_{\text{ext}} - B_{\text{int}})$$
$$= -\frac{2B_{\text{earth}}}{\mu_0} \frac{\mu - 1}{\mu + 1} \cos \phi e_\phi.$$  \hspace{1cm} (A5)

From (3) the applied surface current for a complete degaussing in this case is

$$J_a = \frac{B_{\text{earth}}}{\mu_0} \frac{\mu - 1}{\mu + 1} \cos \phi e_\phi$$

which has the same shape factor $\cos \phi$ as the magnetization surface current given by the above equation. Thus, the numerical scaling method suggested in [4] works for this special circular cylinder case.

References


